· PROGRESSÃO ARITMÉTICA ·

SEQUÊNCIAS

(i) 3 ; 7 ; −1 ; 18 ; 4 ; 15

1º termo 3º termo 6º termo

(ii) 2 ; 5 ; 8 ; 11 ; 14 ; 17

+3 +3 +3 +3 +3

(iii) 20 ; 18 ; 16 ; 14 ; 12 ; 10 ; 8 ...

−2 −2 −2 −2 −2 −2

DEFINIÇÃO

Uma progressão aritmética de razão r é uma sequência tal que:

$$a_n = a_{n-1} + r$$

EXEMPLOS

(i) $\underset{a_1}{2} \xrightarrow{+3} \underset{a_2}{5} \xrightarrow{+3} \underset{a_3}{8} \xrightarrow{+3} \underset{a_4}{11} \xrightarrow{+3} \underset{a_5}{14} \xrightarrow{+3} \underset{a_6}{17}$

$2 ; 5 ; 8 ; 11 ; 14 ; 17$

P.A. de razão $r = 3$

(ii) $\underset{a_1}{20} \xrightarrow{-2} \underset{a_2}{18} \xrightarrow{-2} \underset{a_3}{16} \xrightarrow{-2} \underset{a_4}{14} \xrightarrow{-2} \underset{a_5}{12} \xrightarrow{-2} \underset{a_6}{10} \xrightarrow{-2} \underset{a_7}{8} \ldots$

$20 ; 18 ; 16 ; 14 ; 12 ; 10 ; 8 \ldots$

P.A. de razão $r = -2$

CLASSIFICAÇÃO

$\boxed{r > 0}$ P. A. crescente

EXEMPLO: $-10 \; ; \; -5 \; ; \; 0 \; ; \; 5 \; ; \; 10 \; ; \; 15 \; \ldots$

+5 +5 +5 +5

$\boxed{r < 0}$ P. A. decrescente

EXEMPLO: $12 \; ; \; 9 \; ; \; 6 \; ; \; 3 \; ; \; 0 \; \ldots$

−3 −3 −3

$\boxed{r = 0}$ P. A. estacionária

EXEMPLO: $15 \; ; \; 15 \; ; \; 15 \; ; \; 15$

+0 +0

P.A. como função

Considere a P.A.:

$$\underbrace{2 \; ; \; 7}_{+5} \qquad 2 \xrightarrow{+5} 7 \xrightarrow{+5} 12 \xrightarrow{+5} 17$$

$$2 \; ; \; 7 \; ; \; 12 \; ; \; 17 \; ; \; 22 \; \ldots$$

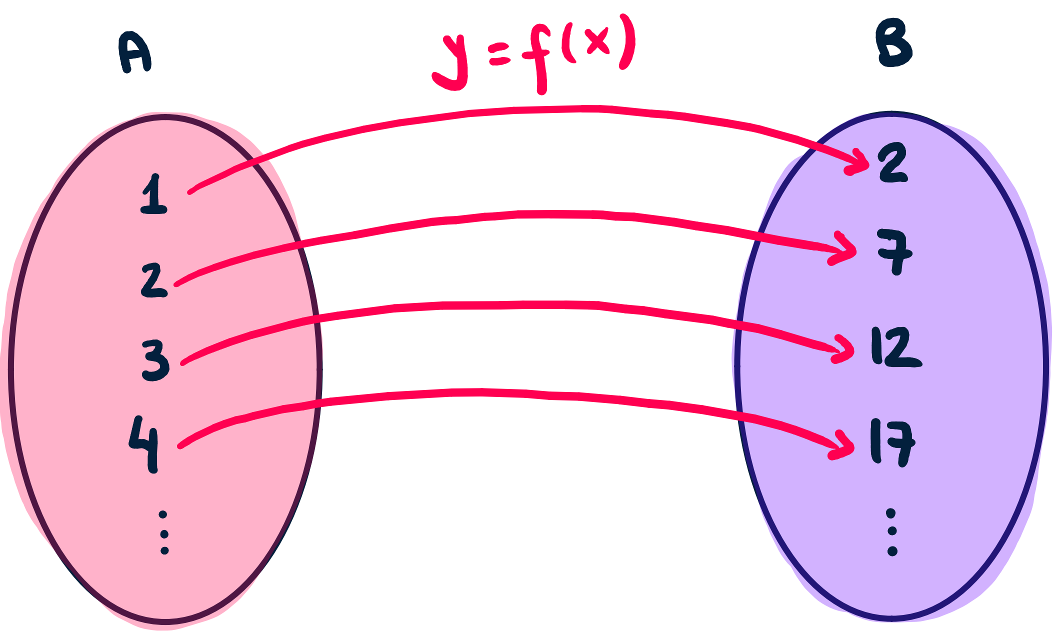
↳ ela pode ser descrita pela função $f(x)$:

$$f(1) = a_1 = 2$$

$$f(2) = a_2 = 7$$

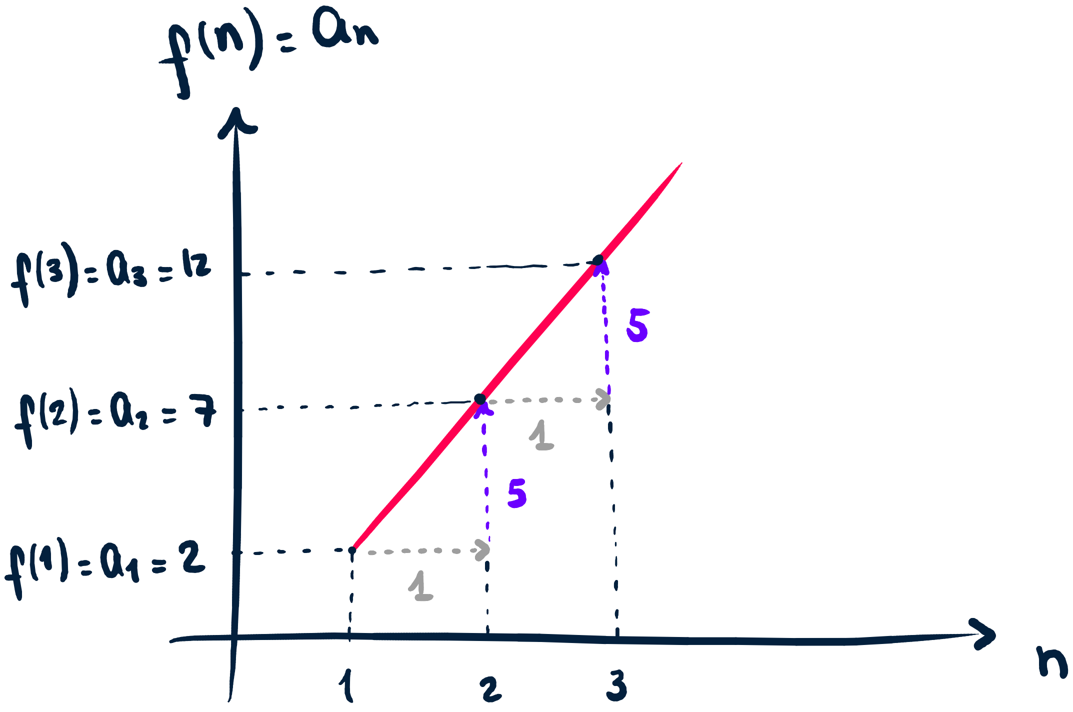
$$f(3) = a_3 = 12$$

$$\vdots$$



DOMÍNIO (x)

CONTRA DOMÍNIO (y)



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TERMO GERAL

P.A. com $a_1 = 3$ e $r = 2$

$$a_1 = 3$$

$$a_2 = 3 + \overbrace{2}^{2-1}$$

$$a_3 = 3 + \overbrace{2 + 2}^{3-1}$$

$$a_4 = 3 + \overbrace{2 + 2 + 2}^{4-1}$$

$$a_5 = 3 + \underbrace{2 + 2 + 2 + 2}_{5-1}$$

$$\vdots$$

$$a_n = 3 + 2 + 2 + \ldots + 2$$

$$= 3 + (n-1) \cdot 2$$

$$= a_1 + (n-1) \cdot r$$

$$\boxed{a_n = a_1 + (n-1) \cdot r}$$

EX.: $3 \; ; \; 8 \; ; \; 13 \; ; \; 18 \; \ldots$

$3 \xrightarrow{+5} 8 \xrightarrow{+5} 13 \xrightarrow{+5} 18$

$17 \cdot 10$

2

(i) $a_{18} = \,?$

$a_n = a_1 + (n-1) \cdot r$

$a_{18} = 3 + (18-1) \cdot 5$

$a_{18} = 3 + 17 \cdot 5$

$\boxed{a_{18} = 88}$

(ii) $a_{51} = \,?$

$a_n = a_1 + (n-1) \cdot r$

$a_{51} = 3 + (51-1) \cdot 5$

$\boxed{a_{51} = 253}$

EXEMPLO 01

Numa P.A. $(-7 \overset{+4}{;} -3 \overset{+4}{;} 1 \dots)$ qual é a posição do termo

a) 293

b) 404

R:

a) $a_n = a_1 + (n-1) \cdot r$

$a_n = -7 + (n-1) \cdot 4$

$a_n = 293 = -7 + (n-1) \cdot 4$

$300 = 4(n-1) \therefore n - 1 = 75 \therefore \boxed{n = 76}$

b) $a_n = -7 + (n-1) \cdot 4 = 404$

$4 \cdot (n-1) = 411 \therefore n - 1 = 102{,}75 \therefore \boxed{n = \cancel{103{,}75}}$

↳ 404 não faz parte dessa P.A.

EXEMPLO 02

Numa P.A. tem-se $a_7 = 24$ e $a_{13} = 72$.

Determine a_4.

R: $a_n = a_1 + (n-1) \cdot r$

i) $a_7 = a_1 + (7-1) \cdot r = 24$

$$a_1 + 6 \cdot r = 24 \quad (I)$$

ii) $a_{13} = a_1 + (13-1) \cdot r = 72$

$$a_1 + 12 \cdot r = 72 \quad (II)$$

$$a_1 + 12 \cdot r = 72 \quad (II)$$

(-)

$$a_1 + 6 \cdot r = 24 \quad (I)$$

$$6r = 48$$

$\boxed{r = 8}$ e $\boxed{a_1 = -24}$

iii) $a_n = a_1 + (n-1) \cdot r$

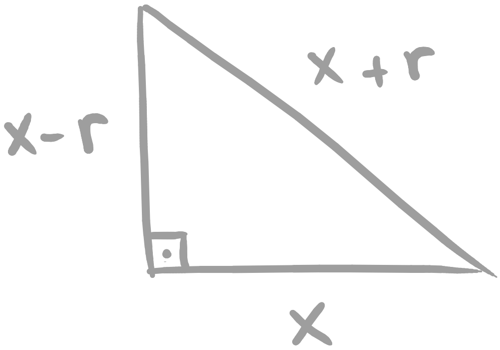
$a_4 = -24 + (4-1) \cdot 8$

$\boxed{a_4 = 0}$

EXEMPLO 03

Os lados de um Δ retângulo formam uma P.A. crescente. Mostre que a razão da P.A. é igual ao raio do círculo inscrito.

R:

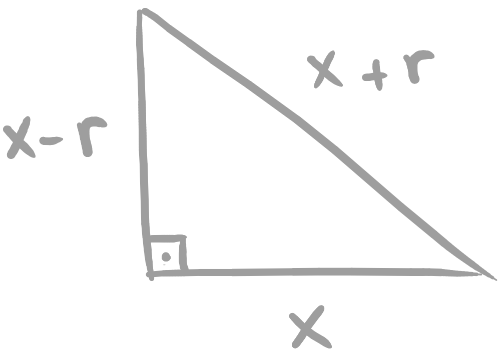


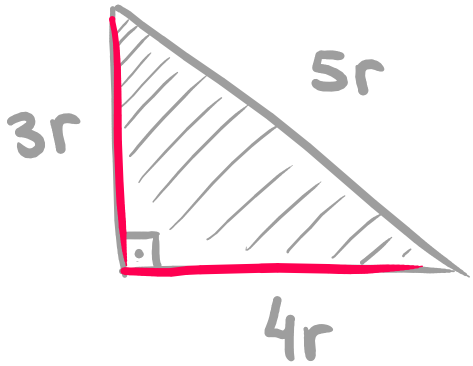
Pitágoras: $(x+r)^2 = (x-r)^2 + x^2$

$$\cancel{x^2} + 2xr + \cancel{r^2} = \cancel{x^2} - 2xr + \cancel{r^2} + x^2$$

$$4r\cdot\cancel{x} = x^{\cancel{2}}$$

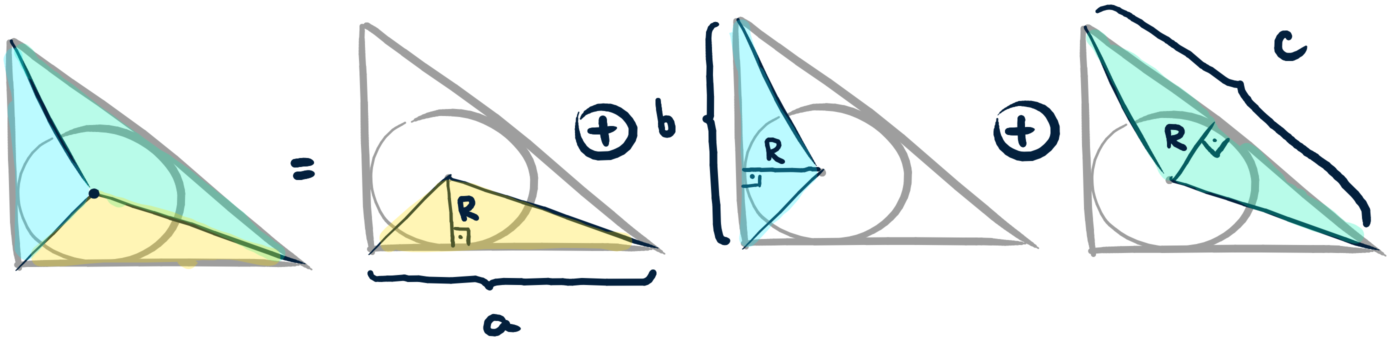
$$x = 4r$$





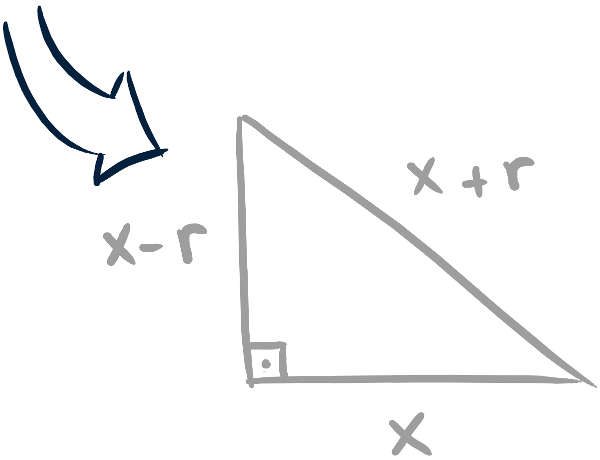
$A = 6r^2$

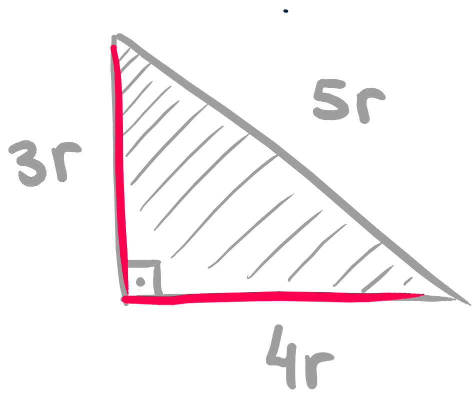
$2p = 12r$



$$A = \frac{a \cdot R}{2} + \frac{b \cdot R}{2} + \frac{R}{2}$$

$$2A = (a+b+c) \cdot R \quad \therefore \quad \boxed{R = \frac{2A}{(a+b+c)}}$$



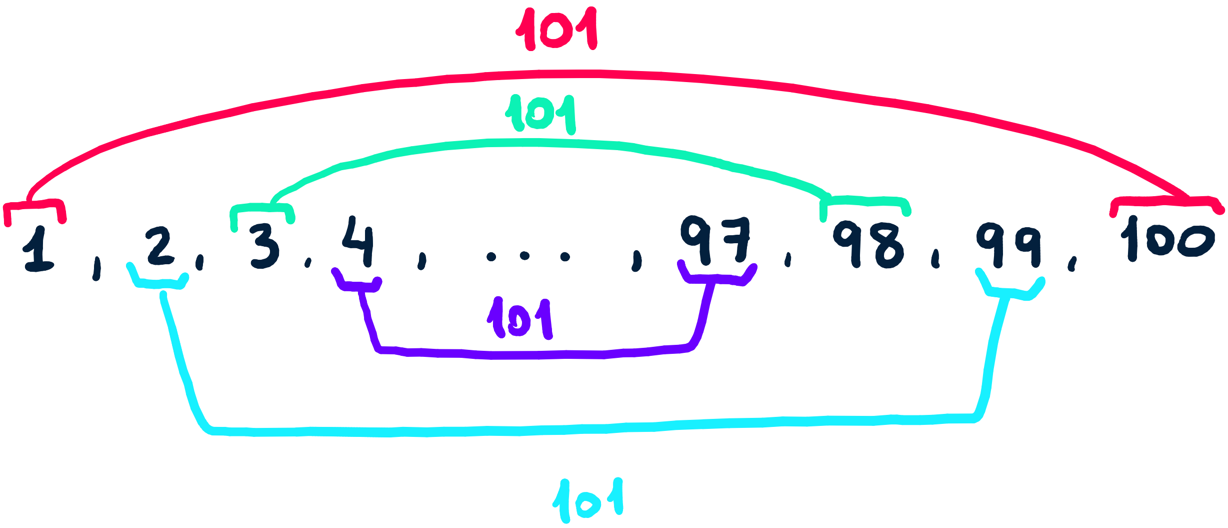


$$R_{insc} = \frac{2A}{(a+b+c)} = \frac{2 \cdot 6r^2}{12 \cdot r} = r$$

SOMA DOS TERMOS

Seja a P.A.:

$$1,\ 2,\ 3,\ 4,\ \ldots,\ 97,\ 98,\ 99,\ 100$$



$$S_{100} = 101 \cdot \frac{100}{2} = 5050$$

$$\boxed{S_n = \frac{(a_1 + a_n) \cdot n}{2}}$$

Demonstração:

$$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n \quad \times(2)$$

$$2S_n = 2a_1 + 2a_2 + \dots + 2a_{n-1} + 2a_n$$

$$2S_n = (a_1 + a_n) + \underbrace{(a_2 + a_{n-1})}_{a_1 + a_n} + \dots + (a_1 + a_n) + \underbrace{(a_2 + a_{n-1})}_{a_1 + a_n} + \dots$$

$$2S_n = (a_1 + a_n) \cdot n$$

$$S_n = \frac{(a_1 + a_n) \cdot n}{2} \quad \blacksquare$$

EXEMPLO 01

Determine a soma dos 20 primeiros termos da sequência $\underbrace{2, 5}_{+3}, \underbrace{5, 8}_{+3} \ldots$

R:

$a_1 = 2$

$r = 3$

(i) $a_n = a_1 + (n-1) \cdot r$

$a_{20} = 2 + (20-1) \cdot 3 = 56$

(ii) $S_n = \dfrac{(a_1 + a_n) \cdot n}{2}$

$S_{20} = \dfrac{(2 + a_{20}) \cdot 20}{2} = \dfrac{(2+56) \cdot 20}{2} = \underline{580}$

EXEMPLO 02

Encontre uma fórmula para a soma dos <u>n</u> primeiros

a) números naturais

b) números ímpares

<u>R</u>:

a) $S_n = 1 + 2 + 3 + 4 + \dots + n$

P.A.

$a_1 = 1$

$r = 1$

$a_n = a_1 + (n-1) \cdot r = \overset{1}{a_1} + n - 1 = n$

$$S_n = \frac{(a_1 + a_n) \cdot n}{2} = \frac{(1+n) \cdot n}{2} = \frac{n \cdot (n+1)}{2}$$

<u>EX</u> $\therefore$ $n = 200$: $S_{200} = \frac{200 \cdot 201}{2} = 20100$

b) ímpares:

$$S_{Imp} = 1 + 3 + 5 + 7 + \dots + n$$

$+2 \quad +2 \quad +2$

P.A.

$a_1 = 1$

$r = 2$

$$a_n = a_1 + (n-1)\cdot r = 1 + (n-1)\cdot 2$$
$$= 1 + 2n - 2$$
$$= 2n - 1$$

$$S_{n_{Imp}} = \frac{(a_1 + a_n)\cdot n}{2} = \frac{(\cancel{1} + 2n - \cancel{1})\cdot n}{2} = n^2$$

$$1 + 3 + 5 + 7 + 9 + \dots + (2n-1) = n^2$$

POTÊNCIAS DOS NATURAIS

(i) $$1 + 2 + 3 + \dots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(ii) $$1^2 + 2^2 + 3^2 + \dots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) $$1^3 + 2^3 + 3^3 + \dots + n^3 = \sum_{k=1}^{n} k^3 = \frac{n^4 + 2n^3 + n^2}{4}$$

Demonstração

(i) P.A. :

$$\underbrace{1+2+3+\dots+n}_{\text{P.A. com } a_1=1 \text{ e } r=1} = \sum_{k=1}^{n} k = \frac{(a_1+a_n)\cdot n}{2}$$

$$= \frac{(1+n)\cdot n}{2}$$

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(ii) $\sum_{K=1}^{n} K^2 = 1^2 + 2^2 + 3^2 + \dots + n^2$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(K+1)^3 = K^3 + 3K^2 + 3K + 1$$

$$\sum_{K=1}^{n} (K+1)^3 = \sum_{K=1}^{n} K^3 + 3\sum_{K=1}^{n} K^2 + 3\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$

- $\sum_{K=1}^{n} (K+1)^3 = \cancel{2^3} + \cancel{3^3} + \cancel{4^3} + 5^3 + \dots + \cancel{n^3} + (n+1)^3$
- $\sum_{K=1}^{n} K^3 = 1^3 + \cancel{2^3} + \cancel{3^3} + \cancel{4^3} + 5^3 + \dots + \cancel{n^3}$
- $\sum_{K=1}^{n} K = \frac{(1+n) \cdot n}{2}$
- $\sum_{K=1}^{n} 1 = 1 + 1 + 1 + 1 + \dots + 1 = n$

$$\sum_{K=1}^{n}(K+1)^3 = \sum_{K=1}^{n}K^3 + 3\sum_{K=1}^{n}K^2 + 3\sum_{K=1}^{n}K + \sum_{K=1}^{n}1$$

$$(n+1)^3 = 1^3 + 3\sum_{K=1}^{n}K^2 + 3\cdot\frac{(1+n)\cdot n}{2} + n$$

$$6\sum_{K=1}^{n}K^2 = 2(n+1)^3 - 2 - 2n - 3n(n+1)$$

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$6\sum_{K=1}^{n}K^2 = 2n^3 + \underline{6n^2} + 6n \cancel{+2} \cancel{-2} - 2n - \underline{3n^2} - 3n$$

$$6\sum_{K=1}^{n}K^2 = 2n^3 + 3n^2 + n$$

$$\sum_{K=1}^{n}K^2 = \frac{(2n^3 + 3n^2 + n)}{6} = \frac{n(n+1)(2n+1)}{6} \quad \square$$

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(iii) $\sum_{K=1}^{n} K^3 = 1^3 + 2^3 + 3^3 + \dots + n^3$

$$(K+1)^4 = K^4 + 4K^3 + 6K^2 + 4K + 1$$

$$\sum_{K=1}^{n} (K+1)^4 = \sum_{K=1}^{n} K^4 + 4\sum_{K=1}^{n} K^3 + 6\sum_{K=1}^{n} K^2 + 4\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$

- $\sum_{K=1}^{n} (K+1)^4 = \cancel{2^4} + \cancel{3^4} + \cancel{4^4} + \dots + \cancel{n^4} + (n+1)^4$

- $\sum_{K=1}^{n} K^4 = 1^4 + \cancel{2^4} + \cancel{3^4} + \cancel{4^4} + \dots + \cancel{n^4}$

- $\sum_{K=1}^{n} K^2 = \dfrac{n(n+1)(2n+1)}{6}$

- $\sum_{K=1}^{n} K = \dfrac{n(n+1)}{2}$

- $\sum_{K=1}^{n} 1 = 1 + 1 + \dots + 1 = n$

$$\sum_{K=1}^{n}(K+1)^4 = \sum_{K=1}^{n} K^4 + 4\sum_{K=1}^{n} K^3 + 6\sum_{K=1}^{n} K^2 + 4\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$

$$(n+1)^4 = 1^4 + 4\sum_{K=1}^{n} K^3 + \cancel{6}\,\frac{n(n+1)(2n+1)}{\cancel{6}} + 4\,\frac{n(n+1)}{2} + n$$

$$4\sum_{K=1}^{n} K^3 = (n+1)^4 - 1^4 - n(n+1)(2n+1) - 2n(n+1) - n$$

$$(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$$

$$4\sum_{K=1}^{n} K^3 = n^4 + \underline{4n^3} + \underline{6n^2} + 4n + \cancel{1} - \cancel{1} - n - \underline{2n^2} - 2n - \underline{2n^3} - \underline{3n^2} - n$$

$$4\sum_{K=1}^{n} K^3 = n^4 + \underline{2n^3} + \underline{n^2}$$

$$\sum_{K=1}^{n} K^3 = \frac{(n^4 + 2n^3 + n^2)}{4} \quad \square$$

Teorema

$$\sum_{K=1}^{n} K^p = 1^p + 2^p + \dots + n^p$$ é um

polinômio de grau $(p+1)$ em n.

Com isso:

$$(ii)\ \sum_{K=1}^{n} K^2 = p(n) = an^3 + bn^2 + cn + d$$

$$\cdot\ p(0) = d = 0$$

$$\cdot\ p(1) = a + b + c = 1$$

$$\cdot\ P(2) = 8a + 4b + 2c = 1^2 + 2^2 = 5$$

$$\cdot\ P(3) = 27a + 9b + 3c = 1^2 + 2^2 + 3^2 = 14$$

↳ Resolvendo o sistema:
$$\begin{cases} a = 1/3 \\ b = 1/2 \\ c = 1/6 \end{cases}$$

$$\sum_{K=1}^{n} K^2 = \frac{(2n^3 + 3n^2 + n)}{6}$$

EXEMPLO 01

Calcule o valor de $S = 1^2 + 2^2 + 3^2 + \dots + 10^2$

R :

$$\sum_{K=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6} = \frac{10\cdot(10+1)(2\cdot 10+1)}{6} = \frac{10\cdot 11\cdot 21}{3\cdot 2}$$

$$= 5\cdot 7\cdot 11 = 5\cdot 77 = \underline{385}\;//$$

EXEMPLO 02

Calcule o valor de

$$S = 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \dots + 100\cdot 101\cdot 102$$

R:

$$S = \sum_{K=1}^{100} K(K+1)(K+2) = \sum_{K=1}^{100} K\,(K^2+2K+K+2)$$

$$S = \sum_{K=1}^{100} (K^3+3K^2+2K) = \sum_{K=1}^{100} K^3 + 3\cdot\sum_{K=1}^{100} K^2 + 2\cdot\sum_{K=1}^{100} K$$

$$= \frac{(n^4+2n^3+n^2)}{4} + 3\,\frac{n(n+1)(2n+1)}{6} + 2\cdot\frac{(1+n)\cdot n}{2}$$

$n=100$

$$= \frac{(100^4+2\cdot 100^3+100^2)}{4} + \frac{100\cdot 101\cdot 201}{2} + 100\cdot 101 =$$

$$= 103.035.150$$