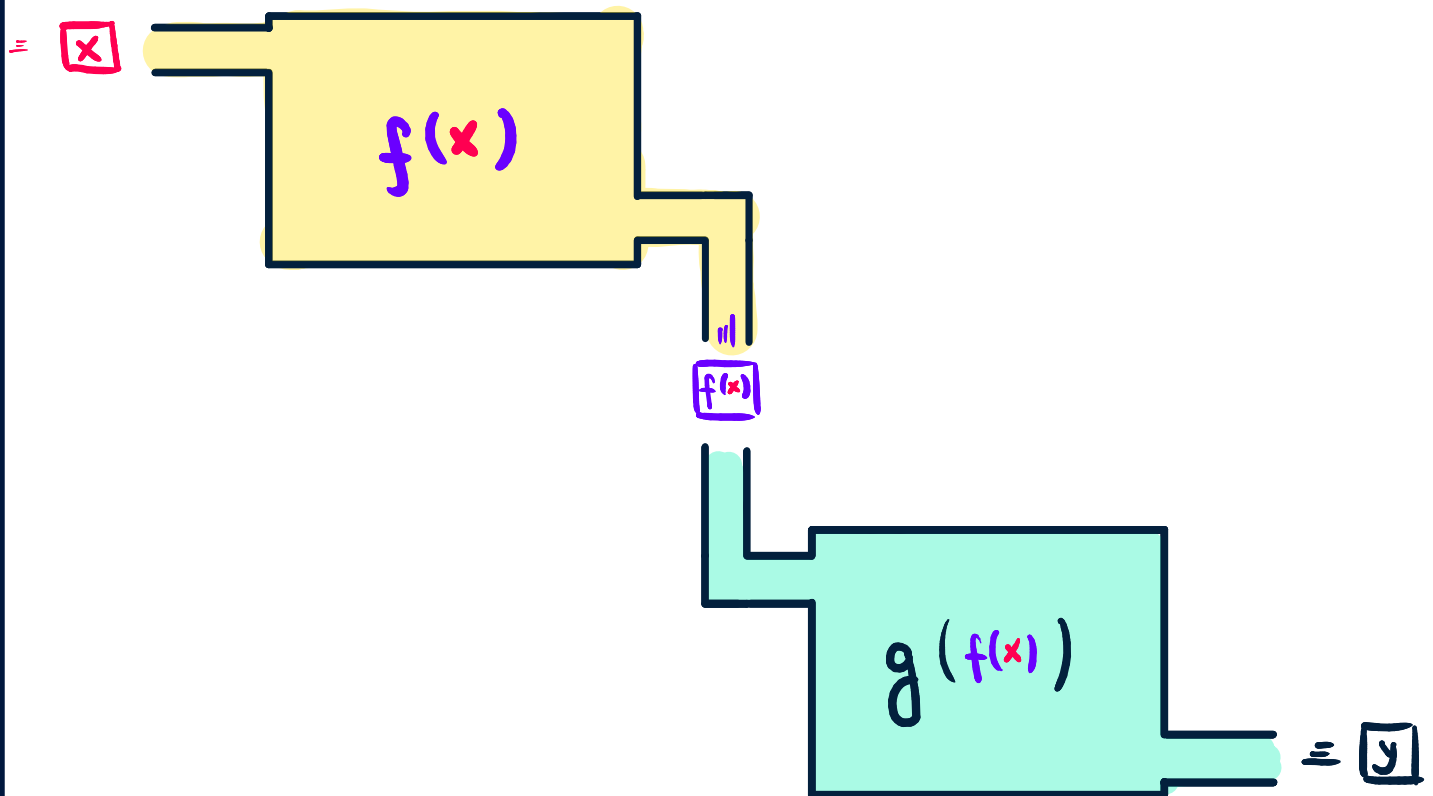


• FUNÇÕES COMPOSTA • INVERSA •

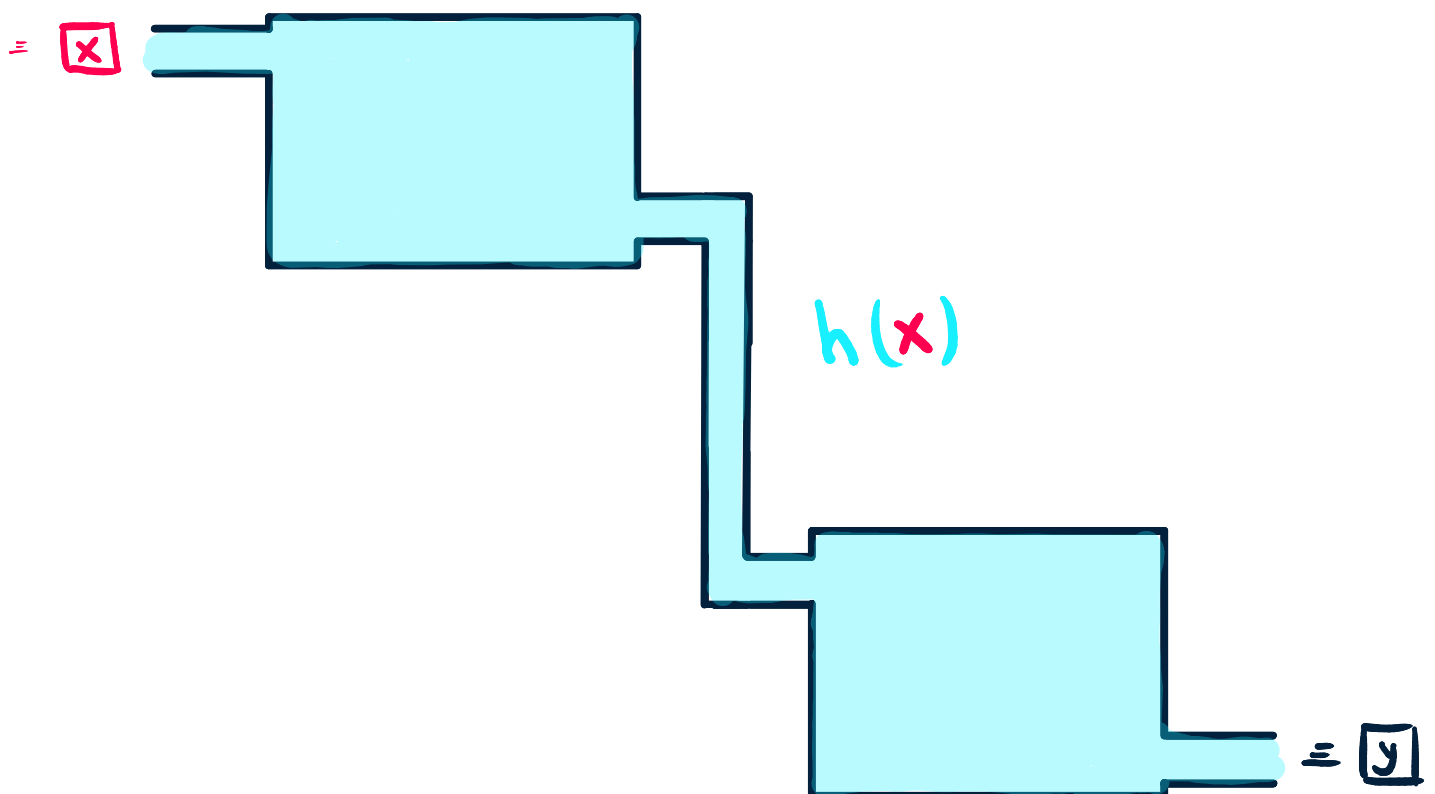
• FUNÇÃO COMPOSTA •





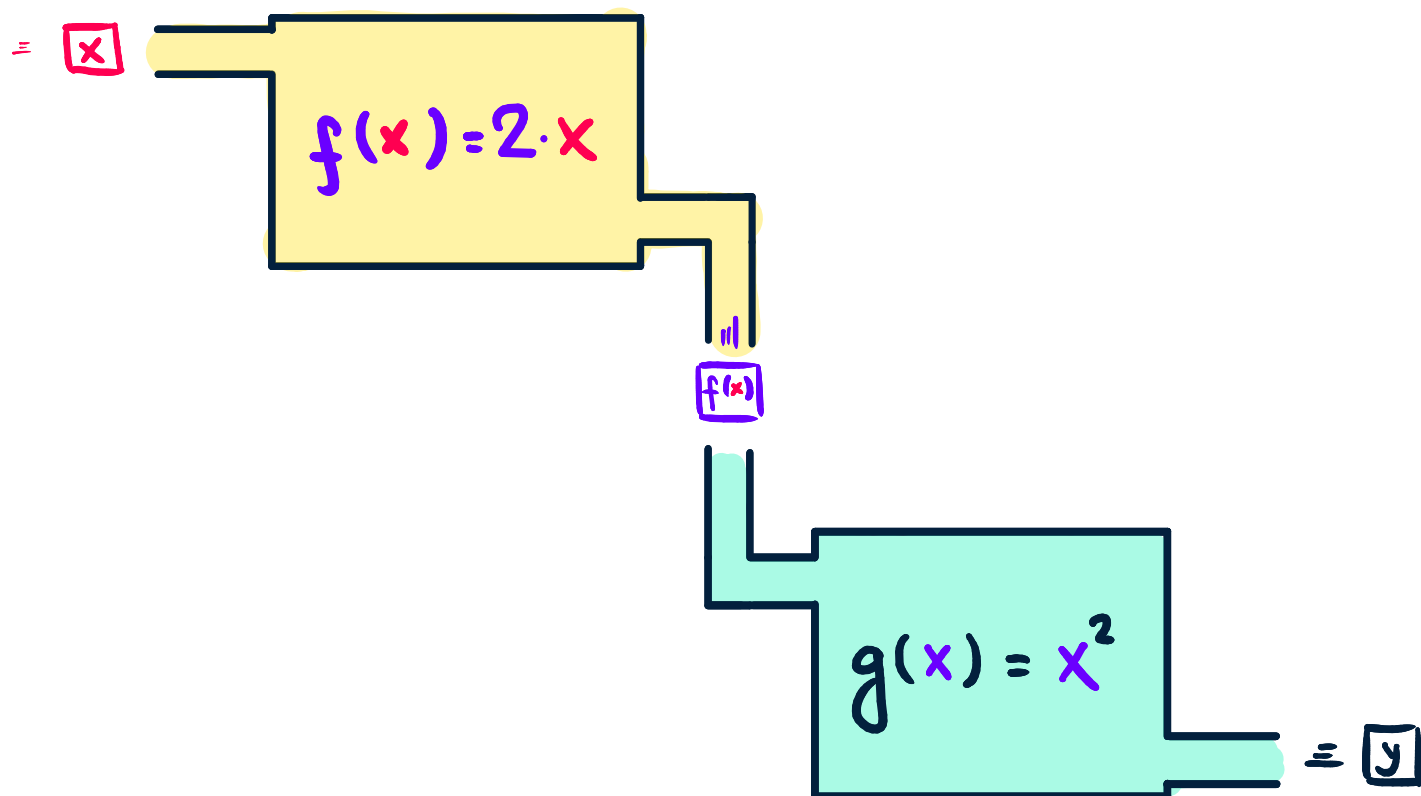
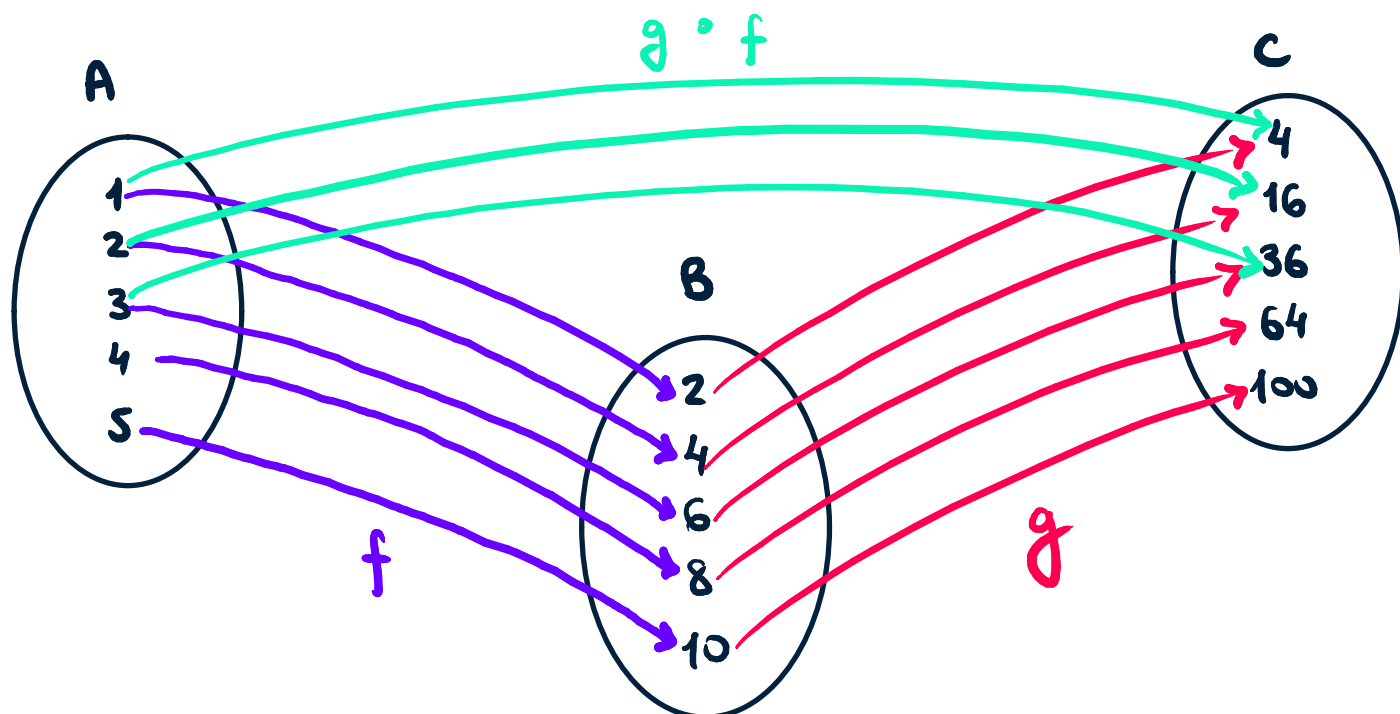
Temos a função composta $h(x)$:

$$h(x) = g(f(x)) = g \circ f$$



$$f(x) = 2x \quad A \rightarrow B \quad \text{e} \quad g(x) = x^2 \quad B \rightarrow C$$

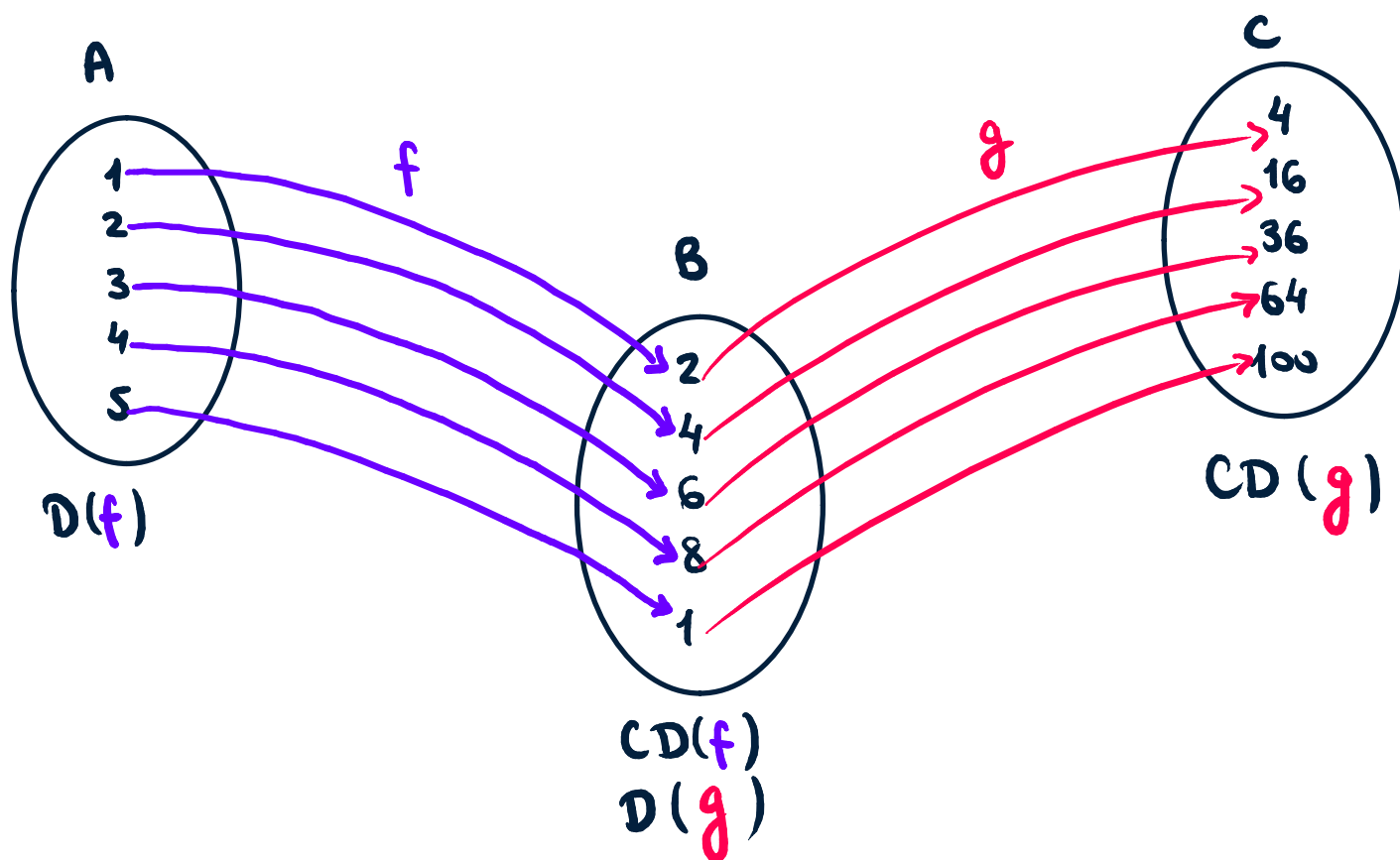
$$g \circ f = g(f(x)) = f(x)^2 = (2x)^2 = 4x^2$$



PROPRIEDADES

(i) Para fazer a composição $g \circ f$ é preciso

que $CD(f) = D(g)$:



(ii) Em geral $g \circ f \neq f \circ g$

Ex. $\therefore f(x) = 2x + 1$ e $g(x) = x^2$

$$\hookrightarrow g \circ f = g(f(x)) = g(2x + 1) = (2x + 1)^2 = 4x^2 + 4x + 1$$

$$\hookrightarrow f \circ g = f(g(x)) = f(x^2) = 2(x^2) + 1 = 2x^2 + 1$$



EXEMPLO 01

Sejam as funções $f(x) = x + 1$ e

$g(x) = x^2 - 2$. Obtenha:

a) A expressão matemática para $h(x) = f(g(x))$

b) $h(0)$ e $h(1)$

c) A expressão matemática para $P(x) = g(f(x))$

d) $P(0)$ e $P(1)$

R :

$$a) h(x) = f \circ g = f(g(x)) = f(x^2 - 2) = x^2 - 2 + 1 = \underline{x^2 - 1}$$

$$b) \begin{aligned} \cdot h(0) &= -1 \\ \cdot h(1) &= 0 \end{aligned}$$

$$g(x) = x^2 - 2$$

$$c) P(x) = g(f(x)) = g(x+1) = (x+1)^2 - 2 = \underline{x^2 + 2x - 1}$$

$$d) \begin{aligned} \cdot P(0) &= -1 \\ \cdot P(1) &= 2 \end{aligned}$$



EXEMPLO 02

Sejam as funções $f(x) = x^3$ e $g(x) = x^4$.

Obtenha:

a) A expressão matemática para $h(x) = f(g(x))$

b) $h(0)$ e $h(1)$

c) A expressão matemática para $P(x) = g(f(x))$

d) $P(0)$ e $P(1)$

R :

a) $h(x) = f(g(x)) = g(x)^3 = (x^4)^3 = x^{12}$

b) $h(0) = 0$
 $h(1) = 1$

$h(x) = P(x)$

c) $P(x) = g(f(x)) = f(x)^4 = (x^3)^4 = x^{12}$

d) $P(0) = 0$
 $P(1) = 1$



EXEMPLO 03

Sejam as funções $f(x) = 3x - 1$ e

$$(g \circ f)(x) = 3x^2 + 4. \text{ Obtenha } g(x).$$

R:

SOLUÇÃO 01

$$g(x) = ax^2 + bx + c$$

$$g(f(x)) = a \cdot f(x)^2 + b \cdot f(x) + c = a(3x-1)^2 + b(3x-1) + c$$

$$= a(9x^2 - 6x + 1) + b \cdot 3x - b + c$$

$$= 9ax^2 - 6ax + a + 3bx - b + c$$

$$= 9ax^2 + (3b - 6a)x + (a - b + c)$$

$$= 3x^2 + 0 \cdot x + 4$$

$$\text{i) } 9a = 3 \therefore a = \frac{1}{3}$$

$$\text{ii) } 3b = 6a \therefore b = \frac{2}{3}$$

$$\text{iii) } c = 4 + b - a \therefore c = \frac{13}{3}$$

$$g(x) = \frac{1}{3}x^2 + \frac{2}{3}x + \frac{13}{3}$$

SOLUÇÃO 02

$$f(x) = 3x - 1 \longrightarrow f(x) + 1 = 3x \longrightarrow x = \frac{f(x) + 1}{3}$$

$$g(f(x)) = 3x^2 + 4$$

$$\hookrightarrow g(f(x)) = 3x^2 + 4 = 3 \cdot \left(\frac{f(x) + 1}{3} \right)^2 + 4$$

$$= \frac{\cancel{3} \cdot (f(x)^2 + 2f(x) + 1)}{\cancel{9} \cancel{3}} + 4$$

$$= \frac{1}{3} f(x)^2 + \frac{2}{3} f(x) + \frac{1}{3} + 4$$

$$= \frac{1}{3} f(x)^2 + \frac{2}{3} f(x) + \frac{13}{3}$$

$$g(f(x)) = \frac{1}{3} f(x)^2 + \frac{2}{3} f(x) + \frac{13}{3}$$

$$g(\square) = \frac{1}{3} \square^2 + \frac{2}{3} \square + \frac{13}{3}$$

$$g(x) = \frac{1}{3} x^2 + \frac{2}{3} x + \frac{13}{3}$$



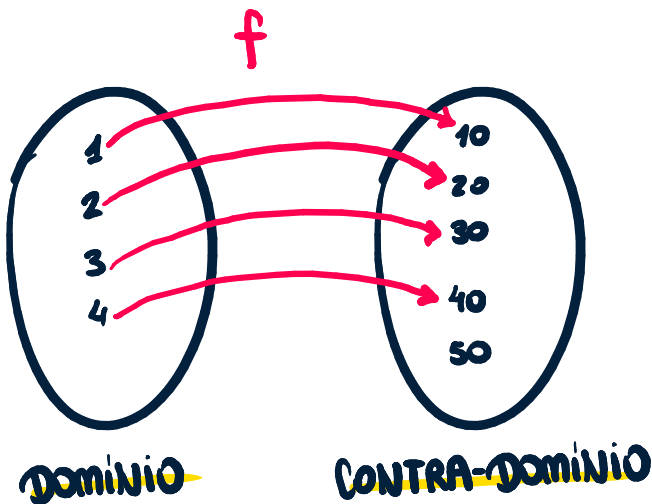
• FUNÇÃO INJETORA •

DEFINIÇÃO

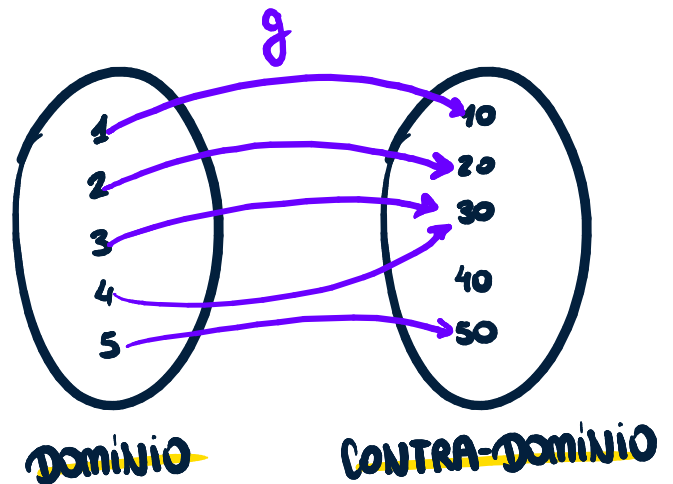
Uma função $f: A \rightarrow B$ é dita injetora se, e somente se, para quaisquer valores x_1 e x_2 de A tivermos $f(x_1) \neq f(x_2)$ se $x_1 \neq x_2$.

EM LINGUAGEM | MATEMÁTICA:

f é injetora $\Leftrightarrow \forall x_1 \in A, \forall x_2 \in A, x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$



f é injetora

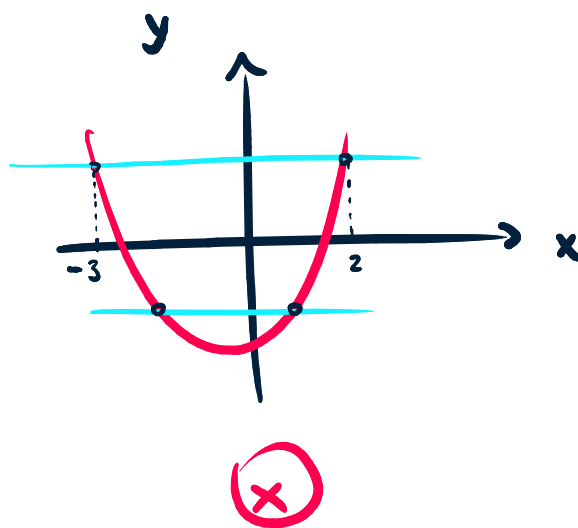
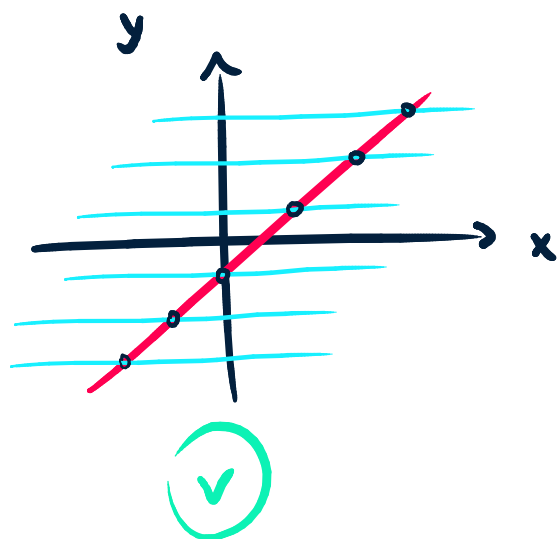
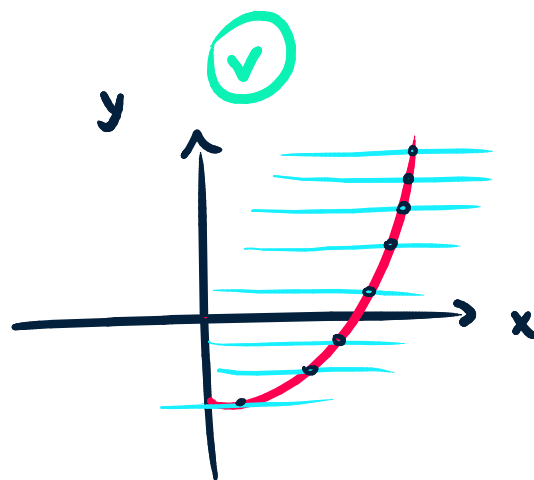
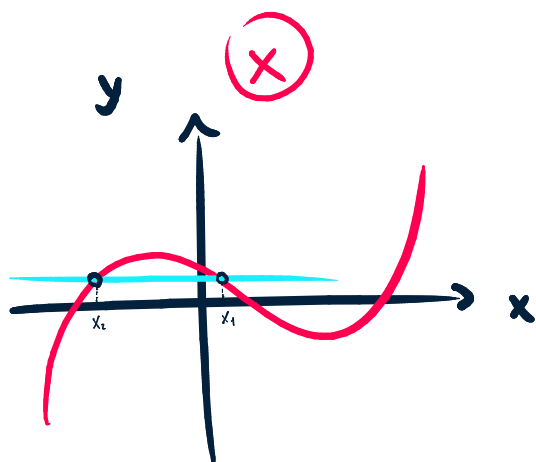


g não é injetora



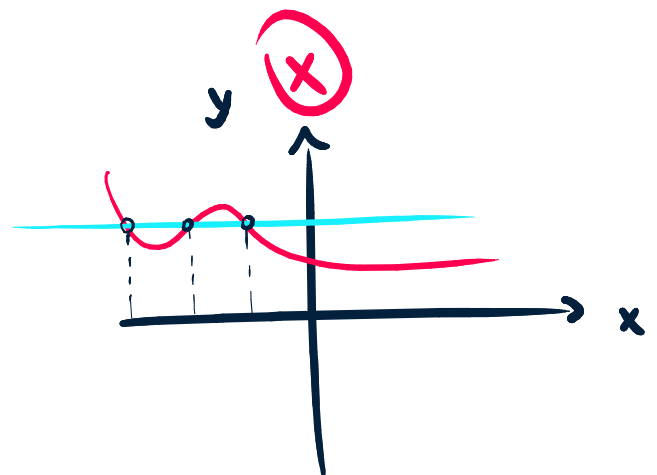
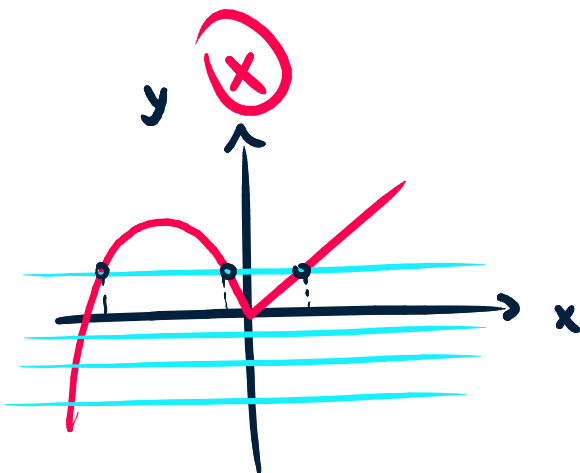
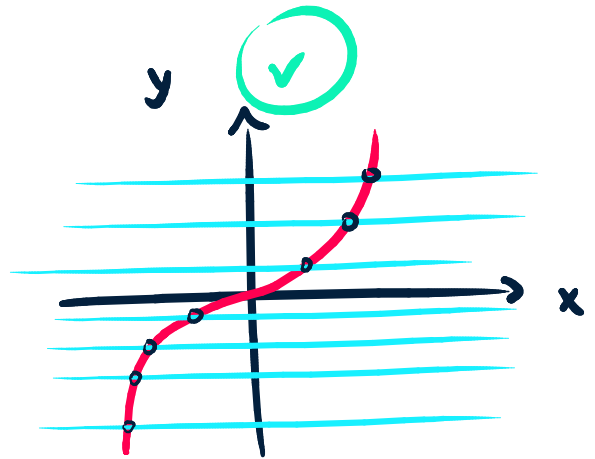
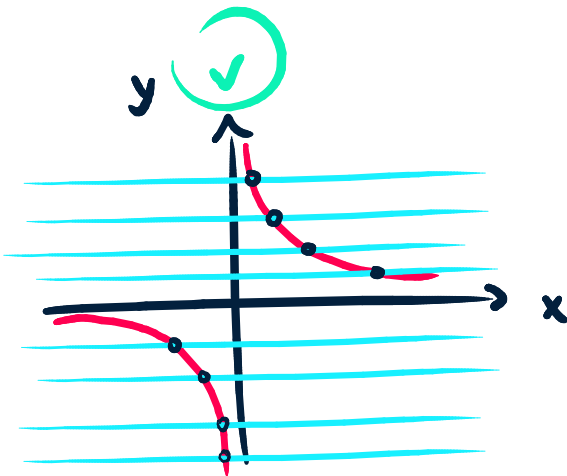
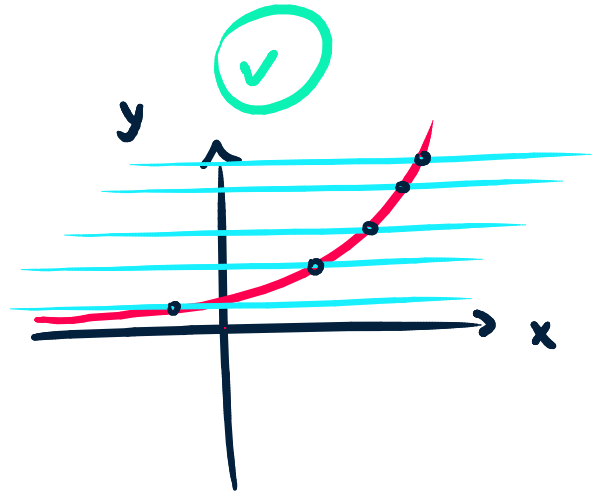
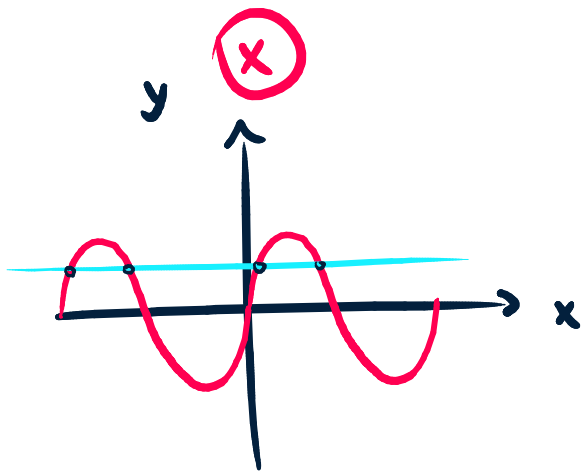
• GRAFICAMENTE •

Se qualquer reta horizontal cortar o gráfico de uma função f em mais de um ponto, então f não é injetora.



EXEMPLO 01

Determine quais são os gráficos de funções injetoras.



EXEMPLO 02

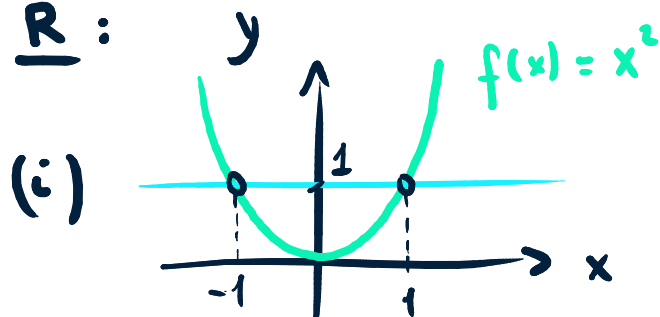
Determine as funções injetoras.

(i) $f: \mathbb{R} \rightarrow \mathbb{R}_+ \mid f(x) = x^2$

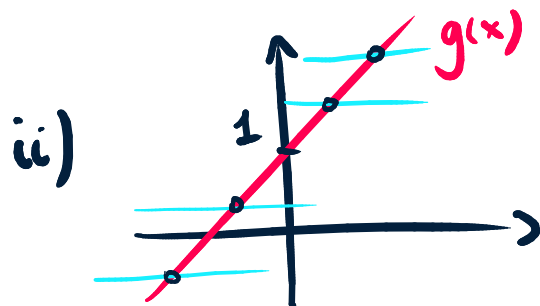
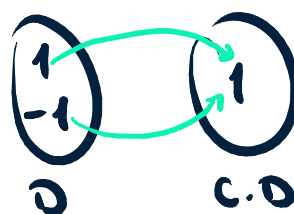
(ii) $g: \mathbb{R} \rightarrow \mathbb{R} \mid g(x) = 2x + 1$

(iii) $h(x): \mathbb{R} \rightarrow \mathbb{R} \mid h(x) = \frac{1}{x^2 + 1}$

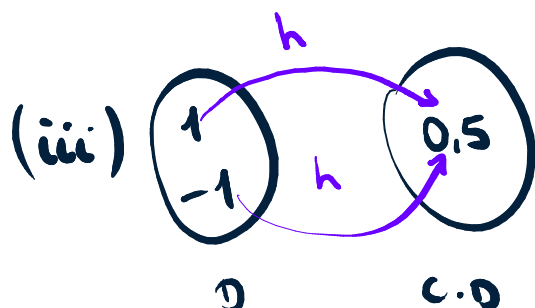
R:



NÃO É INJETORA



É INJETORA



NÃO É INJETORA



• FUNÇÃO SOBREJETORA •

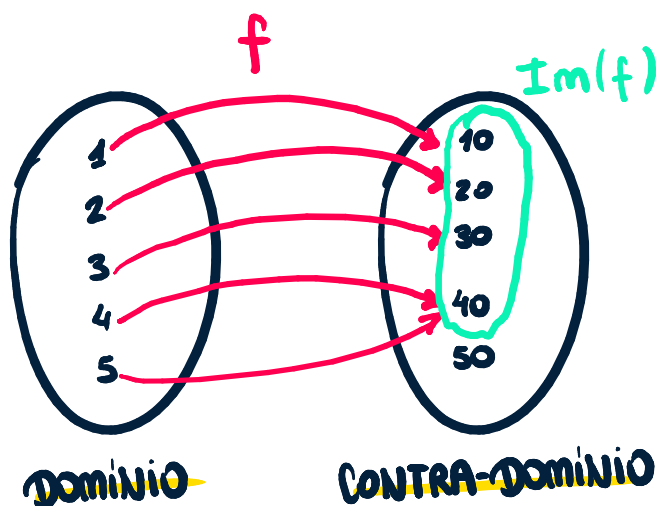
DEFINIÇÃO

Uma função $f: A \rightarrow B$ é dita sobrejetora se, e somente se, para todo y em B existe algum x em A tal que $y = f(x)$.

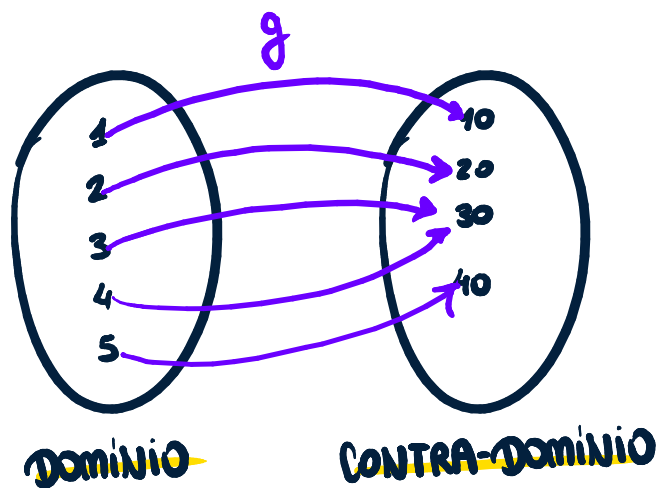
$$\hookrightarrow \text{Im}(f) = B$$

EM LINGUAGEM | MATEMÁTICA:

$$f \text{ é sobrejetora} \iff \forall y \in B \exists x \in A \mid y = f(x)$$



f não é sobrejetora



g é sobrejetora



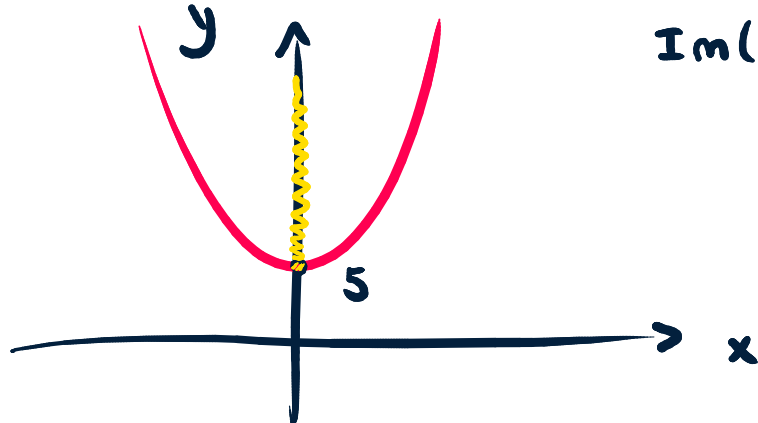
EXEMPLO 01

Determine se a função $f: A \rightarrow B$ com

$$A = \mathbb{R} \text{ e } B = \{y \in \mathbb{R} \mid y \geq 5\} \text{ dada por}$$

$$f(x) = x^2 + 5 \text{ e' sobrejetora.}$$

R:



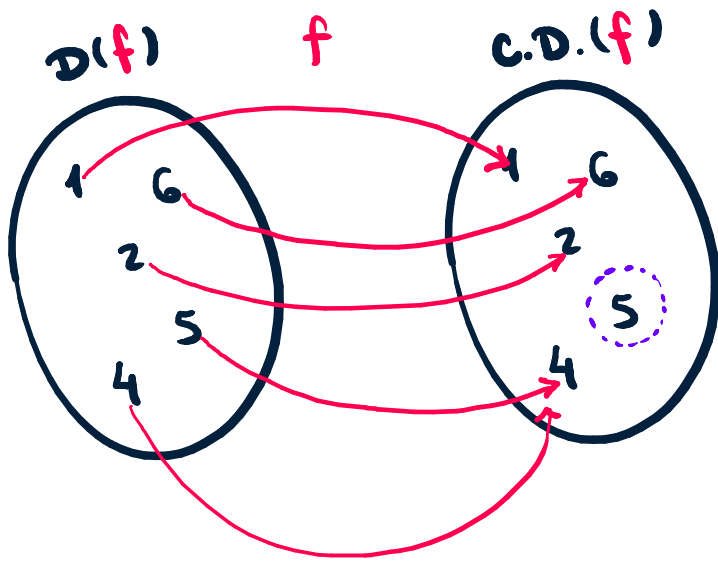
$$\text{Im}(f) = \{y \in \mathbb{R} \mid y \geq 5\}$$

$$\text{C.D.}(f) = \text{Im}(f) : \underline{\text{E' SOBREJETORA}}$$

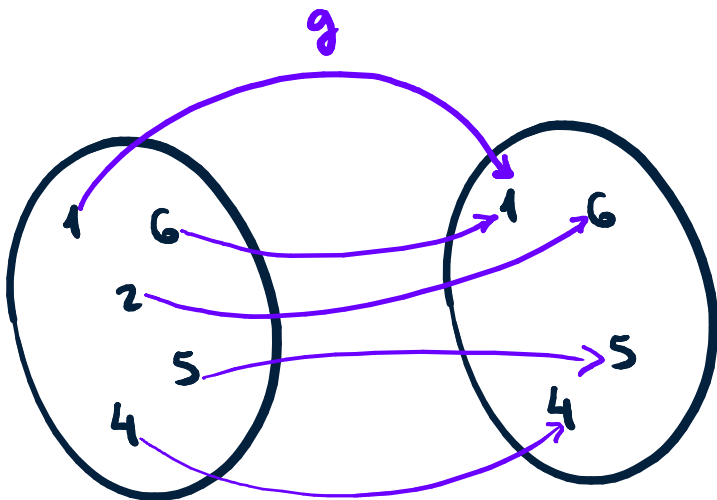


EXEMPLO 02

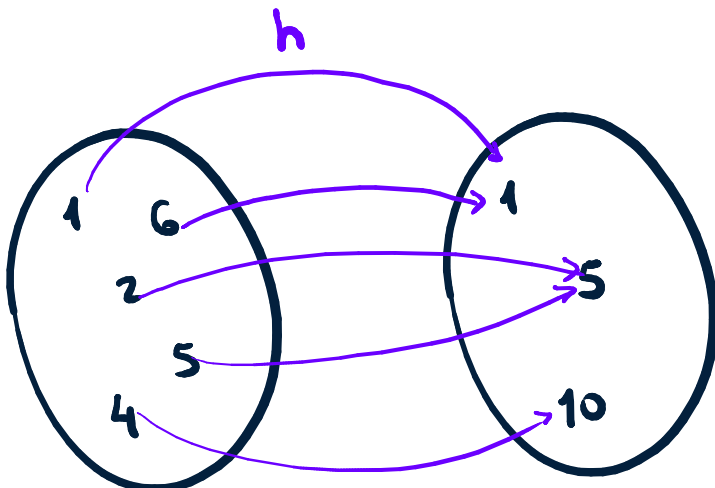
Determine quais funções são sobrejetoras.



$$C.D.(f) \neq Im(f)$$



$$C.D.(g) = Im(g)$$

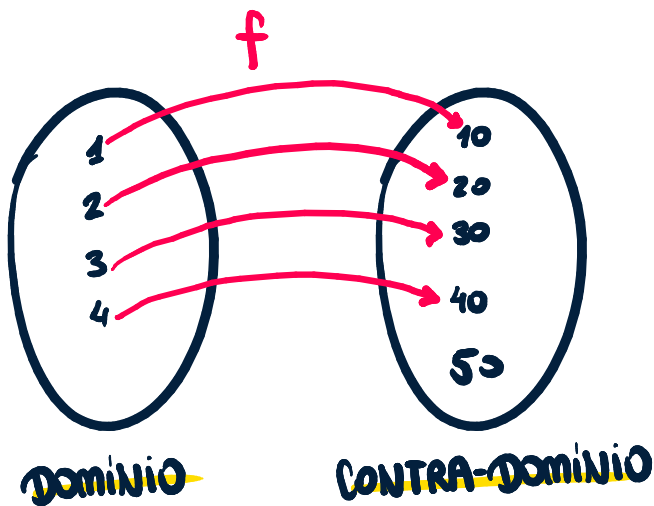


$$C.D.(h) = Im(h)$$



• FUNÇÃO BIJETORA •

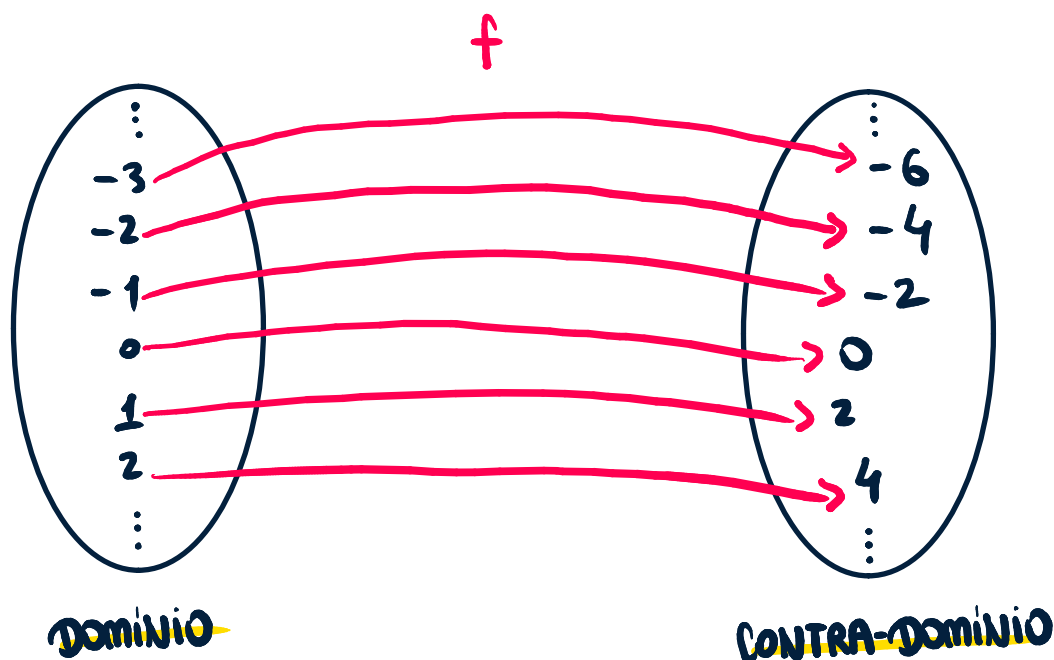
↳ uma função f é bijetora quando é injetora e sobrejetora.



EXEMPLO

A função $f: \mathbb{R} \rightarrow \mathbb{R}$ dada por $f(x) = 2x$ é bijetora?

R :



(i) Se $f(x_1) \neq f(x_2) \Rightarrow \cancel{2x_1} \neq \cancel{2x_2} \Rightarrow x_1 \neq x_2$

↳ É INJETORA

(ii) $D(f) = \mathbb{R} : -\infty < x < +\infty$

$$-\infty < 2x < +\infty$$

$$-\infty < f(x) < +\infty$$

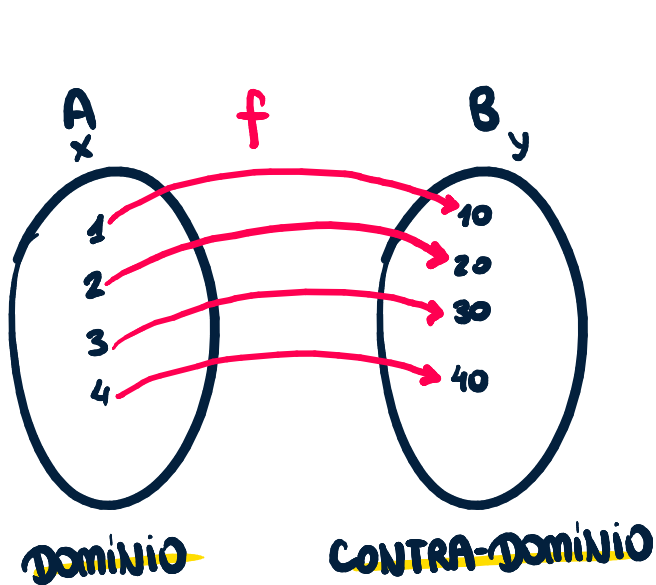
↳ É SOBREJETORA

$$\text{Im}(f) = \mathbb{R} = \text{C.D.}(f)$$

R : SIM

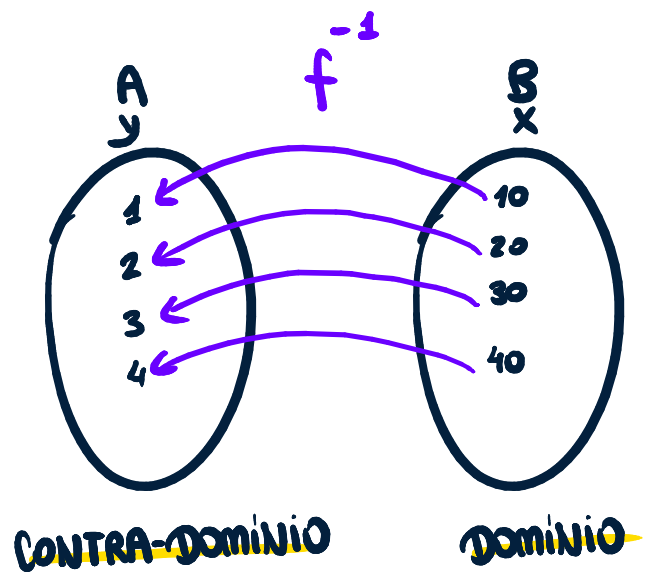


• FUNÇÃO INVERSA •



$$f: A \rightarrow B$$

$$f(x) = 10 \cdot x$$

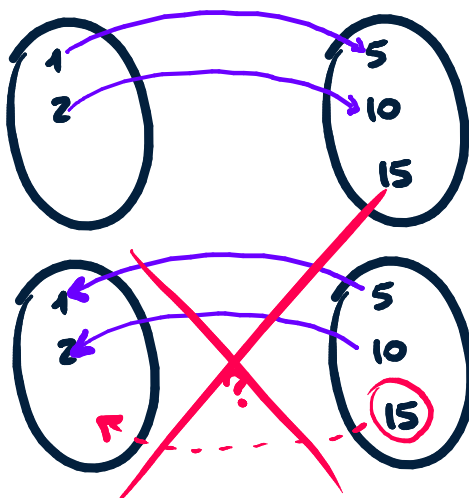


$$f^{-1}: B \rightarrow A$$

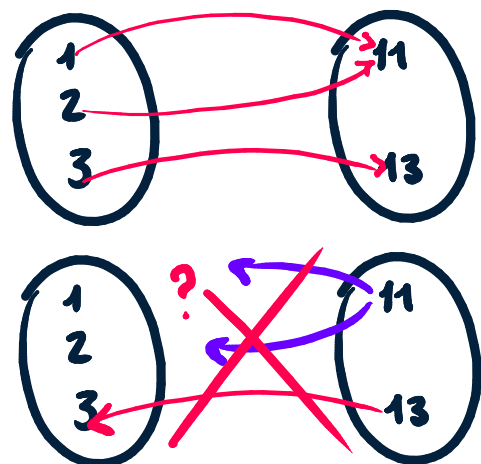
$$f^{-1}(x) = \frac{x}{10}$$

PARA TER UMA INVERSA A FUNÇÃO DEVE SER

(i) **SOBREJETORA**

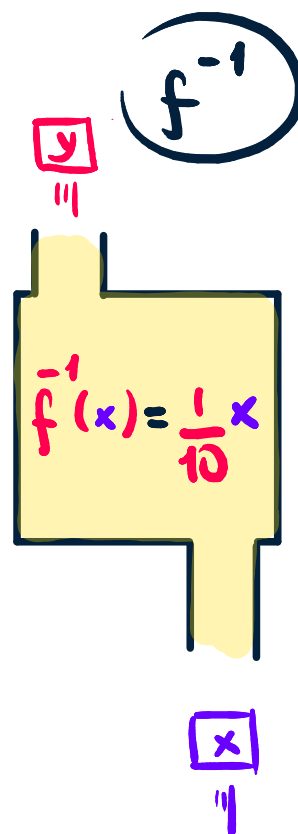
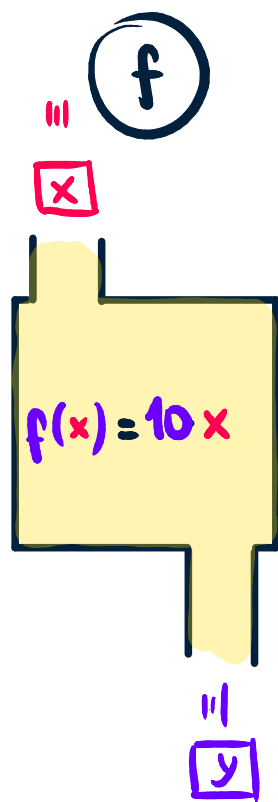


(ii) **INJETORA**



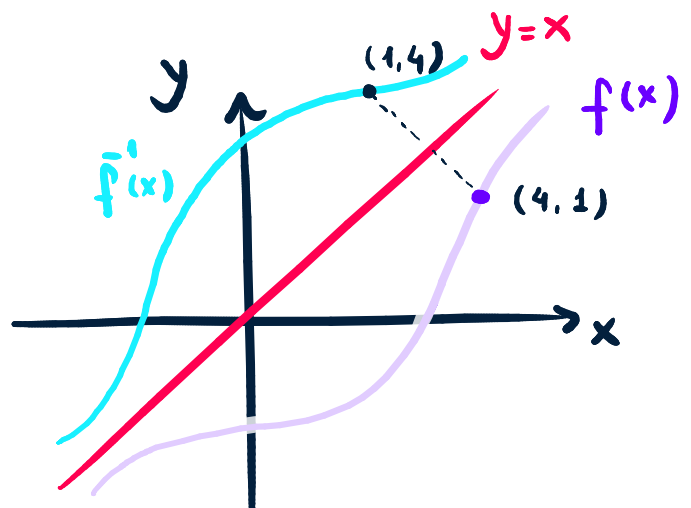
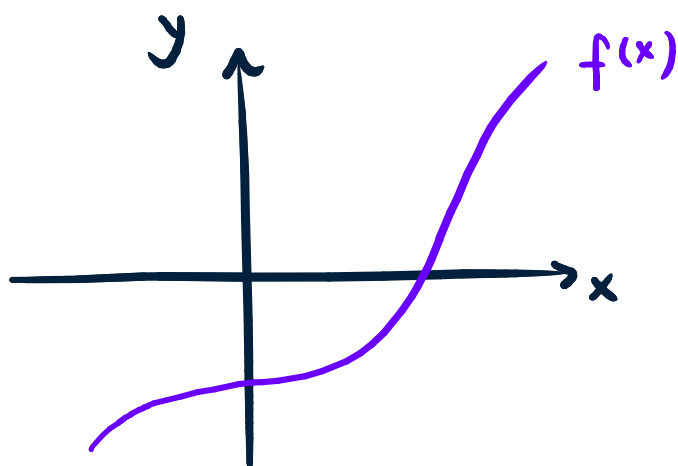
f deve ser bijetora!





PROPRIEDADES DA INVERSA

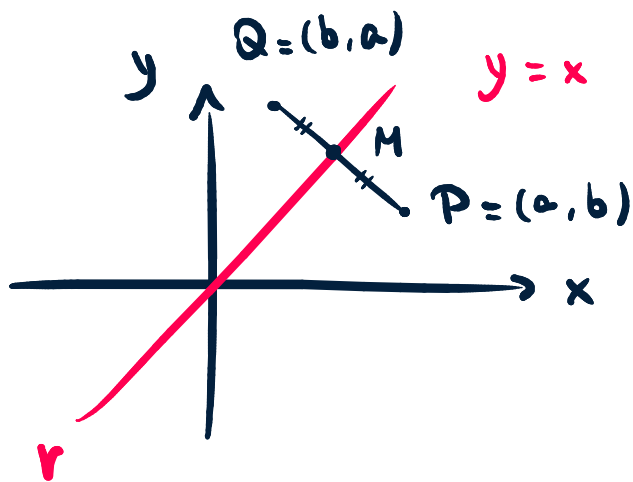
- (i) Só existe f^{-1} se f é bijetora
- (ii) $(x, y) \in f \iff (y, x) \in f^{-1}$
- (iii) $\mathcal{D}(f^{-1}) = \text{Im}(f)$ e $\text{Im}(f^{-1}) = \mathcal{D}(f)$
- (iv) Os gráficos de f e f^{-1} são simétricos
em relação à reta $y = x$.



DEMONSTRAÇÃO

SIMETRIA DOS GRAFICOS DE f e f^{-1}

$$P = (a, b) \in f \iff Q = (b, c) \in f^{-1}$$

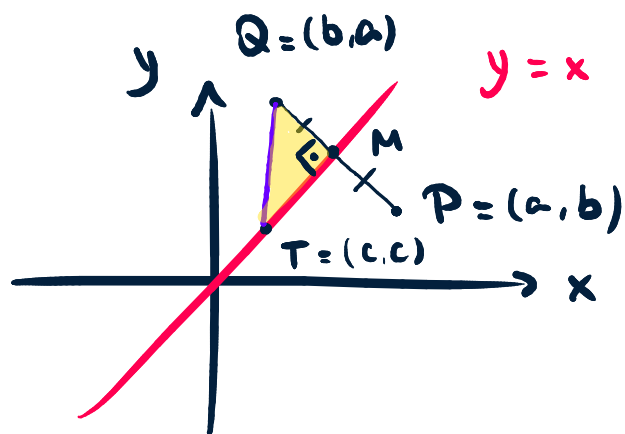


(i) O ponto médio M de $\overline{PQ}$ é:

$$M = \left(\frac{x_P + x_Q}{2}, \frac{y_P + y_Q}{2} \right) = \left(\frac{a+b}{2}, \frac{a+b}{2} \right) \in r$$

(ii)

$$d^2(P, Q) = \Delta x^2 + \Delta y^2$$



$$(MT)^2 = 2 \cdot \left(\frac{a+b-2c}{2} \right)^2$$

$$(MQ)^2 = 2 \cdot \left(\frac{a-b}{2} \right)^2$$

$$T = \begin{pmatrix} c & c \end{pmatrix} \quad (TQ)^2 = (b-c)^2 + (a-c)^2$$

$$Q = \begin{pmatrix} b & a \end{pmatrix}$$



Pitágoras :

$$(TQ)^2 = (MT)^2 + (MQ)^2$$

$$(b-c)^2 + (a-c)^2 = 2 \cdot \frac{(a+b-2c)^2}{4} + 2 \cdot \frac{(a-b)^2}{4}$$

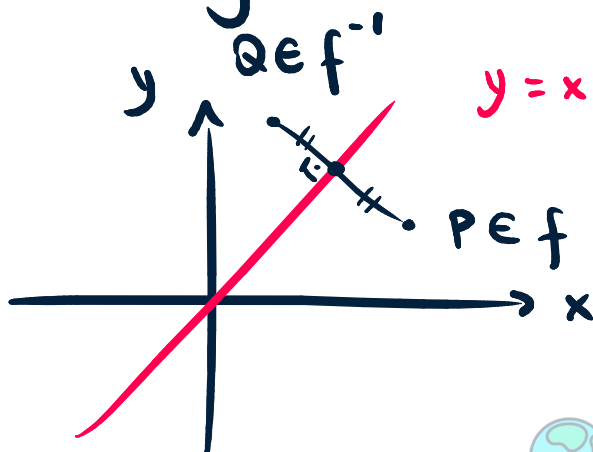
$$2(b^2 - 2bc + c^2 + a^2 - 2ac + c^2) = (a+b-2c)^2 + c^2 - 2cb + b^2$$

$$\underline{2b^2} - 4bc + 4c^2 + \underline{2a^2} - 4ac + 2c^2 = (a+b-2c)^2 + \underline{c^2} - 2cb + \underline{b^2}$$

$$\underline{a^2} + \underline{b^2} + 4c^2 - 4bc - 4ac + 2ab = (a+b-2c)^2$$

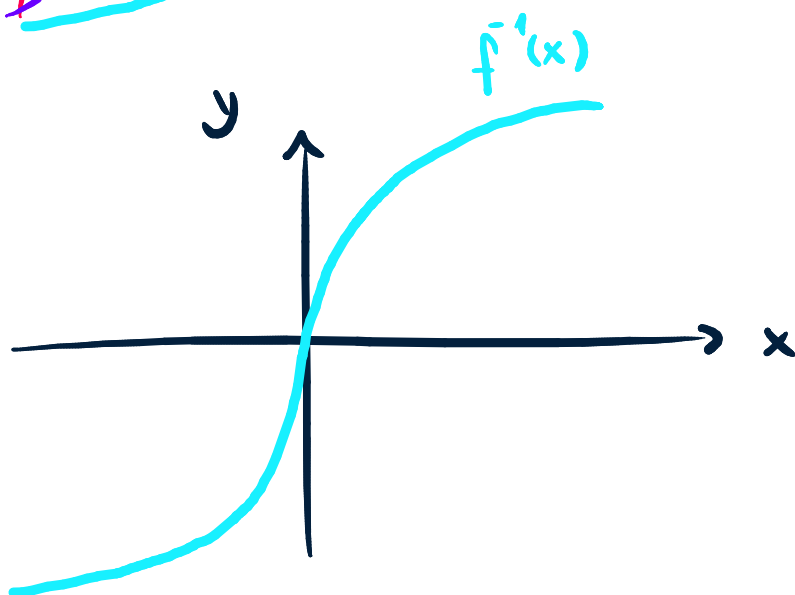
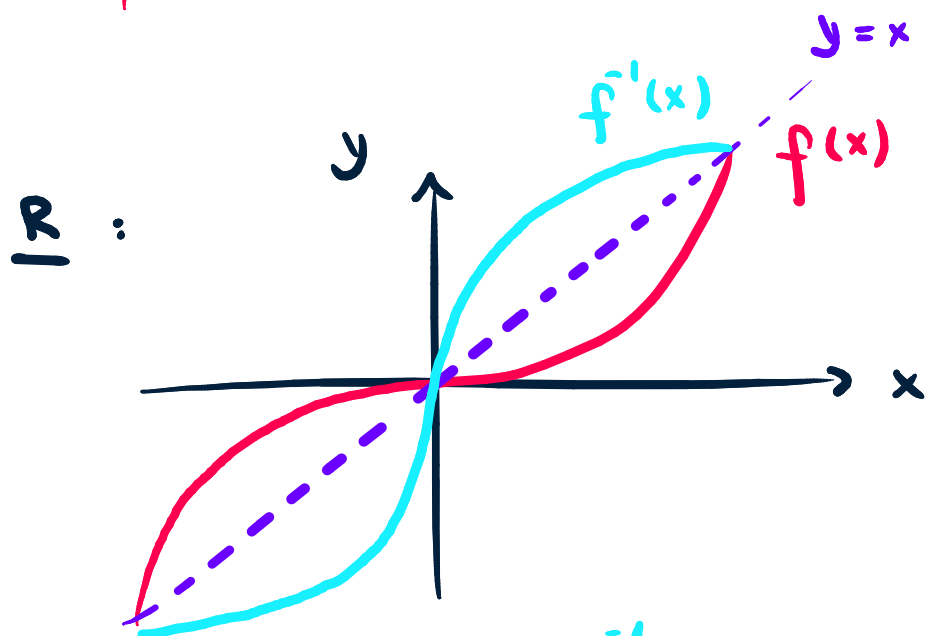
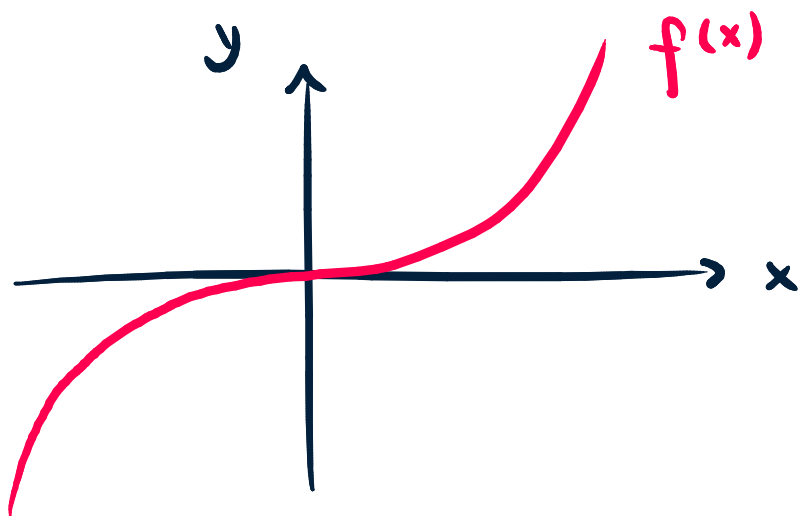
$$\underline{a^2} + \underline{b^2} + \underline{4c^2} - \underline{4bc} - \underline{4ac} + \underline{2ab} = \underline{a^2} + \underline{b^2} + \underline{2ab} - \underline{4ac} - \underline{4bc} + \underline{4c^2}$$

↪ O ângulo é reto!



EXEMPLO 01

Construa o gráfico da inversa de f .



EXEMPLO 02

Encontre a lei da inversa de $f(x) = 3x + 1$ e esboce os gráficos de f e f^{-1} .

R :

$$f(x) = 3x + 1$$

$$\therefore y = 3x + 1 \quad f$$

$$(x, y) \in f$$

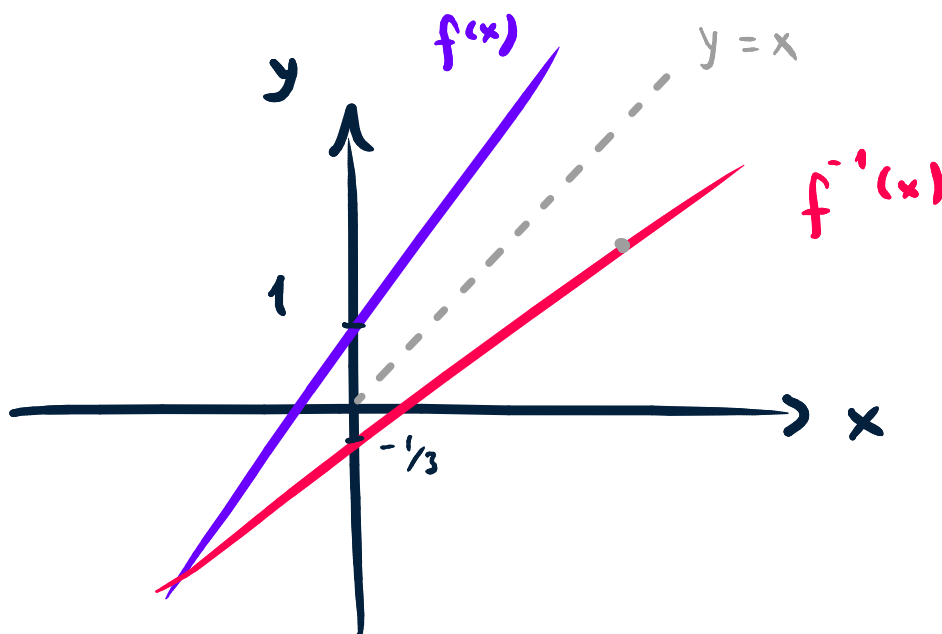
$$(y, x) \in f^{-1}$$

$$x = 3y + 1 \quad f^{-1}$$

$$x - 1 = 3y$$

$$y = \frac{x}{3} - \frac{1}{3}$$

$$\therefore f^{-1}(x) = \frac{x}{3} - \frac{1}{3}$$



EXEMPLO 03

Encontre a lei da inversa de $f(x) = \sqrt{x}$ e esboce os gráficos de f e f^{-1} .

R: $f(x) = \sqrt{x}$

$$f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

$$f^{-1}(x) = x^2$$

$$f^{-1}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

f

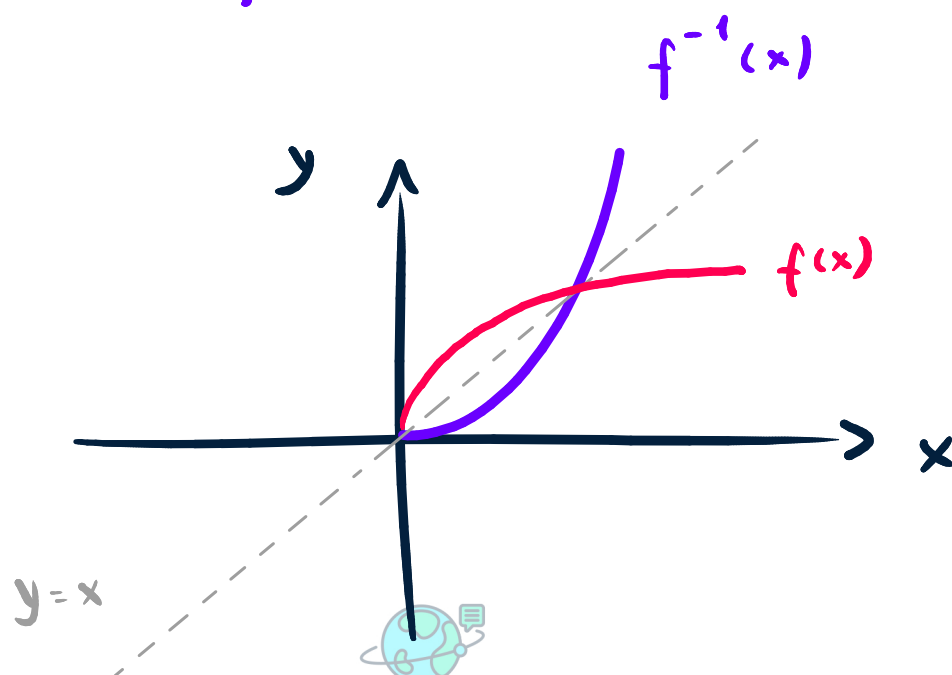
$$y = \sqrt{x}$$

$\longleftrightarrow$
 x, y

$$x = \sqrt{y}$$

$$x^2 = y \therefore y = x^2$$

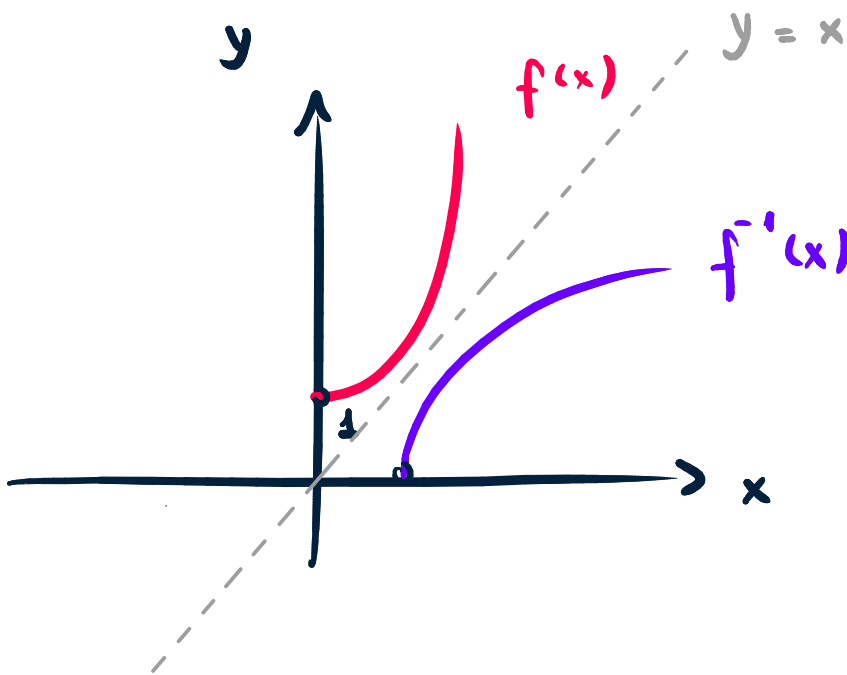
f^{-1}



EXEMPLO 04

Encontre a lei da inversa de $f: \mathbb{R}_+ \rightarrow B$ com $B = \{y \in \mathbb{R} \mid y \geq 1\}$ dada por $f(x) = x^2 + 1$

R:



$$f: y = x^2 + 1$$

$$f^{-1}: x = y^2 + 1 \quad \therefore \quad y^2 = x - 1 \quad \therefore \quad y = +\sqrt{x-1}$$

$$f: \mathbb{R}_+ \rightarrow B$$

$$f^{-1}: B \rightarrow \mathbb{R}_+$$

$$\boxed{f^{-1}(x) = \sqrt{x-1}}$$